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# Maximal avoidable sets of words(Algebraic Theory of Codes and Related Topics)

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# Maximal avoidable sets of words

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We use the following notations.

$\Sigma$  : an alphabet (a finite set of letters),

$\Sigma^*$  : the set of words over  $\Sigma$ ,

$\Sigma^\omega$  : the set of infinite words (sequences),

$\Sigma^\# := \Sigma^* \cup \Sigma^\omega$ ,

$\Sigma^+ := \Sigma^* - \{1\}$ , where 1 is the empty word.

For  $x = a_1 a_2 \dots$ ,  $y = b_1 b_2 \dots \in \Sigma^\#$  define the distance of  $x$  and  $y$  by

$$d(x, y) = \frac{1}{\min\{n \mid a_n \neq b_n\}}.$$

As is well-known ([4]),  $(\Sigma^\#, d)$  is a compact totally disconnected metric space.

Let  $x, y \in \Sigma^*$  and  $X \subset \Sigma^\#$ . We say  $y$  avoids  $x$ , if  $y$  does not contain  $x$  as subword, and  $y$  avoids  $X$ , if  $y$  avoids every  $x$  in  $X$ .  $X$  is called avoidable, if there is an infinite word  $y$  avoiding  $X$ , otherwise  $X$  is called unavoidable. Avoidability of sets of words called patterns were studied in [1].

Example 1. Let  $X = \{v^2 \mid v \in \Sigma^+\}$ . Then  $y$  avoids  $X$  if and only if  $y$  is square-free. It is a famous fact that  $X$  is avoidable if  $|\Sigma| \geq 3$

([5]).

An avoidable set  $X$  is maximal, if any set properly containing  $X$  is unavoidable.

Theorem 1. For any avoidable set  $X$ , there is a maximal avoidable set containing  $X$ .

For a given  $X \subset \Sigma^*$ , the set

$$\text{Min}(X) = \{x \in X \mid \text{any } x' \in X \text{ is not a proper subword of } x\}$$

is called the base of  $X$ . Easily we see  $y$  avoids  $X$  if and only if  $y$  avoids  $\text{Min}(X)$ .  $X$  is finitely based if  $\text{Min}(X)$  is a finite set. The base of a maximal avoidable set is called a critical set of words.

Corollary. For any avoidable set  $X$  which is factor-free, that is, any word in  $X$  is not a subword of another word in  $X$ , there is a critical set containing  $X$ .

Example 2. Let  $\Sigma = \{a, b\}$ . Then,  $\{a^2, ab, ba\}$  and  $\{a^2, b^2\}$  are critical sets.

An infinite word  $x$  is recurrent, if for any subword  $v$  of  $x$ , there is an integer  $k(v) > 0$  such that any subword of length  $k$  of  $x$  contains  $v$  as subword. In this situation  $v$  is said to be recurrent in  $x$ . If  $x = v^\omega$  for some  $v \in \Sigma^*$ ,  $x$  is called periodic; the shortest such  $v$  is the period of  $x$ . A periodic infinite word is recurrent, but the converse is not true.

The shift transformation  $\tau$  is a mapping from  $\Sigma^\omega$  to itself defined by

$$\tau(x) = a_2 a_3 \cdots \quad \text{for } x = a_1 a_2 \cdots.$$

Obviously,  $\tau$  is a surjective continuous mapping.

A subshift  $S$  is a non-empty closed subset of  $\Sigma^\omega$  invariant under  $\tau$ .  $S$  is minimal, if it does not contain a subshift properly. For a given set  $X \subset \Sigma^*$  of words,  $S(X)$  is the set of infinite words avoiding  $X$ . For a given subshift  $S \subset \Sigma^\omega$ ,  $X(S)$  is the set of words which do not appear as subwords of elements of  $S$ .

Theorem 2. For an avoidable set  $X$ ,  $S(X)$  is a subshift. If  $X$  is maximal, then  $S(X)$  is minimal. Conversely, if  $S$  is a subshift, then  $X(S)$  is an avoidable set. If  $S$  is minimal, then  $X(S)$  is maximal. This gives a 1-1 correspondence between maximal avoidable sets and minimal subshifts.

Lemma 1. An avoidable set  $X$  is maximal if and only if any word out of  $X$  is recurrent in any infinite word avoiding  $X$ .

Theorem 3 (Morse-Hedlund [4]).  $S \subset \Sigma^\omega$  is a minimal subshift if and only if

$$S = \overline{\{ \tau^n(x) \mid n = 0, 1, 2, \dots \}}$$

for some recurrent infinite word  $x$ . Moreover,

(1)  $S$  is perfect, if  $x$  is non-periodic. In this case every element in  $S$  is non-periodic.

(2)  $S$  is finite, if  $x$  is periodic. In this case every element in  $S$  is periodic.

Corollary (c.f. [3, Theorem 4.2]). Let  $X$  be an avoidable set such that for any  $v \in \Sigma^+$ ,  $v^n$  does not avoid  $X$  for  $x \gg 0$ , then  $S(X)$  contains a perfect subset

Theorem 4. Let  $X$  be a maximal avoidable set. Then,  $S(X)$  is finite if and only if  $X$  is finitely based.

For an avoidable set  $X$ , the radical  $\text{rad}(X)$  of  $X$  is the intersection of all the maximal avoidable sets containing  $X$ .  $X$  is called reduced, if  $X = \text{rad}(X)$ .

Lemma 2. A word  $v$  is in  $\text{rad}(X)$ , if and only if any recurrent infinite word avoiding  $X$  avoids  $v$ .

Corollary. Any word out of  $\text{rad}(X)$  is extensible to a recurrent infinite word avoiding  $X$ .

Theorem 5. If  $X$  is a reduced avoidable set, then every isolated point of  $S(X)$  is periodic.

Corollary. If  $X$  is a reduced avoidable set such that for any  $v \in \Sigma^+$ ,  $v^n$  does not avoid  $X$  for  $n \gg 0$ , then  $S(X)$  is perfect.

A set  $X$  of words is quasi-maximal, if  $\text{rad}(X)$  is maximal.

Theorem 6. Let  $X$  be an avoidable set. Then following statements are equivalent.

(1)  $X$  is quasi-maximal.

(2)  $S(X)$  contains a unique minimal subshift.

(3) For any  $n > 0$ , there is a word  $v$  of length  $n$  such that  $X \cup \{v\}$  is unavoidable.

(4) For any word  $w$  such that  $X \cup \{w\}$  is unavoidable and for any  $n > 0$ , there is a word  $v$  of length  $n$  such that  $X \cup \{wv\}$  is unavoidable.

Example 3. Let  $X = \{a^2, bab\} \subset \{a, b\}^*$ . Then,  $b^\omega$  and  $ab^\omega$  are only infinite words avoiding  $X$ , and  $X \cup \{b^n\}$  is unavoidable for any  $n > 0$ .

Thus  $X$  is quasi-maximal.

An unavoidable set  $X$  is said to be minimal, if  $X - \{v\}$  is avoidable for any  $v \in X$ . As is easily seen ([2]), a minimal unavoidable set is finite.

Conjecture I (Ehrenfeucht, see [2]). For any unavoidable set  $X$ , there is a word  $x \in X$  and a letter  $a \in \Sigma$  such that  $(X - \{x\}) \cup \{xa\}$  is unavoidable.

Conjecture II. For any minimal unavoidable set  $X$ , there is a word  $x$  in  $X$  such that  $X - \{x\}$  is a quasi-maximal avoidable set.

Theorem 7. Conjecture I and Conjecture II are equivalent.

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